

# ON AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH FIXED MEAN VALUE<sup>1</sup>

Alexander G. Babenko

Krasovskii Institute of Mathematics and Mechanics,  
Ural Branch of the Russian Academy of Sciences;  
Institute of Mathematics and Computer Science, Ural Federal University,  
Ekaterinburg, Russia,  
babenko@imm.uran.ru

**Abstract:** Let  $T_n^+$  be the set of nonnegative trigonometric polynomials  $\tau_n$  of degree  $n$  that are strictly positive at zero. For  $0 \leq \alpha \leq 2\pi/(n+2)$ , we find the minimum of the mean value of polynomial  $(\cos \alpha - \cos x)\tau_n(x)/\tau_n(0)$  over  $\tau_n \in T_n^+$  on the period  $[-\pi, \pi]$ .

**Key words:** Trigonometric polynomials, Extremal problem.

Let  $T_n$  be the space of trigonometric polynomials of degree  $n$  with real coefficients, and let  $T_n^+$  be the set of nonnegative polynomials from  $T_n$  that are strictly positive at zero. For a real  $\alpha$  we define

$$\chi_n(\alpha) = \inf_{\tau_n \in T_n^+} \frac{1}{2\pi\tau_n(0)} \int_{-\pi}^{\pi} \tau_n(x)(\cos \alpha - \cos x) dx. \quad (1)$$

In 1915, Fejér [4] (see also [2, vol. 2, Sec. 6, Problem 52]) proved the following statement.

**Fejér's Theorem.** Let the polynomial  $\tau_n(x) = a_0 + \sum_{\nu=1}^n (a_\nu \cos \nu x + b_\nu \sin \nu x)$  belong to the set  $T_n^+$ . Then

$$\sqrt{a_1^2 + b_1^2} \leq 2a_0 \cos \frac{\pi}{n+2}. \quad (2)$$

This inequality turns into the equality for the polynomial

$$t_n(x) = \left( \cos \frac{n+2}{2} x \right)^2 / \left( \cos x - \cos \frac{\pi}{n+2} \right)^2. \quad (3)$$

This theorem is equivalent to the statement that

$$\chi_n(\pi/(n+2)) = 0. \quad (4)$$

For  $0 \leq \alpha < \pi$ , put

$$Q_{(n+3)/2, \alpha}(x) = \left( \sin \frac{n+1}{2} \alpha \sin \frac{n+3}{2} x - \sin \frac{n+3}{2} \alpha \sin \frac{n+1}{2} x \right) / \sin \frac{\alpha}{2}, \quad 0 < \alpha < \pi, \quad (5)$$

$$Q_{(n+3)/2, 0}(x) = \lim_{\alpha \rightarrow 0} Q_{(n+3)/2, \alpha}(x) = (n+1) \sin \frac{n+3}{2} x - (n+3) \sin \frac{n+1}{2} x.$$

In this paper we prove the following result.

<sup>1</sup>The paper was originally published in a hard accessible collection of articles *Approximation of Functions by Polynomials and Splines* (The Ural Scientific Center of the Academy of Sciences of the USSR, Sverdlovsk, 1985), p. 15–22 (in Russian).

**Theorem.** Let  $n$  be a nonnegative integer and  $0 \leq \alpha \leq 2\pi/(n+2)$ . Then (1) takes the value

$$\chi_n(\alpha) = \frac{\left(\sin \frac{n+3}{2}\alpha - \sin \frac{n+1}{2}\alpha\right)(1 - \cos \alpha)}{(n+3)\sin \frac{n+1}{2}\alpha - (n+1)\sin \frac{n+3}{2}\alpha}, \quad 0 < \alpha \leq \frac{2\pi}{n+2}, \quad (6)$$

$$\chi_n(0) = \lim_{\alpha \rightarrow 0} \chi_n(\alpha) = \frac{6}{(n+1)(n+2)(n+3)},$$

and the infimum is attained for the polynomial

$$\tau_{n,\alpha}(x) = \left( \frac{Q_{(n+3)/2,\alpha}(x)}{(\cos x - \cos \alpha) \sin(x/2)} \right)^2, \quad (7)$$

where  $Q_{(n+3)/2,\alpha}$  is given by (5).

Note that  $\chi_n(\alpha) \geq 0$  for  $0 \leq \alpha \leq \pi/(n+2)$  and  $\chi_n(\alpha) \leq 0$  for  $\pi/(n+2) \leq \alpha \leq 2\pi/(n+2)$ . First we prove two auxiliary statements. Set  $\alpha_0 = \pi$ ,  $\alpha_1 = 2\pi/3$ , and for  $n \geq 2$  let  $\alpha_n$  be the first positive root of the equation

$$\left(\sin \frac{n+3}{2}x\right) / \sin \frac{n+1}{2}x = c_n, \quad c_{2m} = -1, \quad c_{2m-1} = -\frac{m+1}{m}. \quad (8)$$

It is easy to see that for  $r \geq 2$  we have

$$\alpha_{2r-2} = \pi/r, \quad 2\pi/(2r+1) < \alpha_{2r-1} < \pi/r. \quad (9)$$

**Lemma 1.** If  $n$  is a nonnegative integer and  $0 < \alpha < \alpha_n$ , then the function  $Q_{(n+3)/2,\alpha}$  defined by (5) has exactly  $[(n+5)/2]$  zeros  $x_0 = 0 < x_1 = \alpha < x_2 < x_3 < \dots < x_{[(n+3)/2]}$  in the interval  $[0, \pi]$ . For each polynomial  $\tau_{n+1} \in T_{n+1}$  we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \tau_{n+1}(x) dx &= \frac{\sin \frac{n+1}{2}\alpha - \sin \frac{n+3}{2}\alpha}{(n+3)\sin \frac{n+1}{2}\alpha - (n+1)\sin \frac{n+3}{2}\alpha} \tau_n(0) \\ &+ \sum_{k=1}^{[(n+3)/2]} g_{n+1}(x_k) (\tau_{n+1}(x_k) + \tau_{n+1}(-x_k)), \end{aligned} \quad (10)$$

where

$$\begin{aligned} g_{2r-1}(x) &= \frac{\sin x}{2r \sin x - \sin 2rx}, \\ g_{2r}(x) &= \begin{cases} \frac{\sin x}{2(r \sin x - \sin rx \cos(r+1)x)}, & x \neq \pi, \\ \frac{\sin r\alpha + \sin(r+1)\alpha}{4(r \sin(r+1)\alpha + (r+1) \sin r\alpha)}, & x = \pi. \end{cases} \end{aligned} \quad (11)$$

Moreover, the numbers  $(\frac{2\pi}{n+2} - \alpha)g_{n+1}(x_{[(n+3)/2]})$ ,  $g_{n+1}(x_k)$ ,  $1 \leq k \leq [(n+1)/2]$ , are nonnegative.

**P r o o f.** First we consider the case when  $n = 0$  and  $0 < \alpha < \pi$ . The function  $Q_{3/2,\alpha}(x) = 2\sin(x/2)(\cos x - \cos \alpha)$  has two zeros  $x_0 = 0$ ,  $x_1 = \alpha$  in the interval  $[0, \pi]$ . We have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tau_1(x) dx = \frac{\cos \alpha}{\cos \alpha - 1} \tau_1(0) + \frac{\tau_1(\alpha) + \tau_1(-\alpha)}{2(1 - \cos \alpha)}, \quad \tau_1 \in T_1,$$

since this formula is valid for the polynomials 1,  $\sin x$ ,  $\cos x$  and thus the lemma follows for  $n = 0$ .

Now let  $n = 1$  and  $0 < \alpha < 2\pi/3$ . Then the function  $Q_{2,\alpha}(x) = 4\cos \frac{\alpha}{2} \sin x(\cos x - \cos \alpha)$  has three zeros  $x_0 = 0$ ,  $x_1 = \alpha$ ,  $x_2 = \pi$  in the interval  $[0, \pi]$ . The quadrature formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tau_2(x) dx = \frac{1 - 2\cos \alpha}{4(1 - \cos \alpha)} \tau_2(0) + \frac{\tau_2(\alpha) + \tau_2(-\alpha)}{2(1 - \cos \alpha)} + \frac{1 + 2\cos \alpha}{8(1 + \cos \alpha)} (\tau_2(\pi) + \tau_2(-\pi)), \quad \tau_2 \in T_2,$$

holds, for it holds for the polynomials  $\sin x$ ,  $\sin 2x$ ,  $(1 + \cos x)(\cos \alpha - \cos x)$ ,  $1 - \cos 2x$ ,  $(1 - \cos x)(\cos \alpha - \cos x)$  which generate the space  $T_2$ . This proves the lemma for  $n = 1$ .

Next we consider the case of an odd  $n = 2r - 1$ ,  $r \geq 2$ , and  $0 < \alpha < \alpha_{2r-1}$ . The function (5) can be written in the form  $Q_{r+1,\alpha}(x) = (\cos x - \cos \alpha) \sin x S_{r-1,\alpha}(x) / \sin(\alpha/2)$ , where

$$S_{r-1,\alpha}(x) = \frac{\sin r\alpha \sin(r+1)x - \sin(r+1)\alpha \sin rx}{(\cos x - \cos \alpha) \sin x} \quad (12)$$

is a cosine polynomial of degree  $r - 1$ . To study the zeros of the polynomial  $S_{r-1,\alpha}$ , we write it in the form  $S_{r-1,\alpha}(x) = \frac{(f(x) - f(\alpha)) \sin r\alpha \sin rx}{\sin x(\cos x - \cos \alpha)}$ , where  $f(x) = \frac{\sin(r+1)x}{\sin rx}$ . When  $x$  runs over the intervals  $(0, \pi/r)$ ,  $((r-1)\pi/r, \pi)$ , and  $(k\pi/r, (k+1)\pi/r)$ ,  $1 \leq k \leq r-2$ , then the values of  $f$  run continuously over the intervals  $((r+1)/r, -\infty)$ ,  $(+\infty, -(r+1)/r)$ , and  $(+\infty, -\infty)$ , respectively. Thus, taking into account the definition (8) of  $\alpha_{2r-1}$ , we see that for each  $\alpha$  in the interval  $(0, \alpha_{2r-1})$  the polynomial  $S_{r-1,\alpha}$  has exactly  $r - 1$  zeros  $x_2 < x_3 < \dots < x_r$  in the interval  $(\alpha, \pi)$ . Moreover, these zeros are all simple since  $S_{r-1,\alpha}$  has degree  $r - 1$ . It is known [3, p. 403, formulae 30, 31, 33] that

$$\frac{1}{\pi} \int_0^\pi \frac{\sin mx \cos \nu x}{\sin x} dx = \begin{cases} 1, & m > \nu, m + \nu = 2k - 1; \\ 0, & m > \nu, m + \nu = 2k; \\ 0, & m \leq \nu. \end{cases} \quad (13)$$

It follows that for the polynomial (12) we have

$$\frac{1}{\pi} \int_0^\pi S_{r-1,\alpha}(x) \cos \nu x (1 + \cos x)(\cos \alpha - \cos x) dx = \sin(r+1)\alpha - \sin r\alpha, \quad \nu = 0, 1, \dots, r-1.$$

Consequently, for each cosine polynomial  $C_{r-1}$  of degree  $r - 1$  we have

$$\frac{1}{\pi} \int_0^\pi S_{r-1,\alpha}(x) C_{r-1}(x) (1 + \cos x)(\cos \alpha - \cos x) dx = (\sin(r+1)\alpha - \sin r\alpha) C_{r-1}(0). \quad (14)$$

Thus, the polynomial  $S_{r-1,\alpha}$  is orthogonal with the weight  $(\cos x - \cos \alpha)(1 - \cos x)(1 + \cos x)$  to all cosine polynomials of degree  $r - 2$ .

We will need the following known result (e.g., [1, pp. 162, 163]). Let the weight  $v(x)$  and the points  $a_1, \dots, a_m$  in the interval  $[0, \pi]$  be given. A quadrature formula of the form

$$\int_0^\pi C_{2\nu+m-1}(x) v(x) dx = \sum_{\ell=1}^m A_\ell C_{2\nu+m-1}(a_\ell) + \sum_{k=1}^\nu B_k C_{2\nu+m-1}(x_k)$$

which is exact for cosine polynomials of degree  $2\nu + m - 1$  exists if and only if there exists a cosine polynomial  $S_\nu$  of degree  $\nu$  which is orthogonal to all cosine polynomials of degree  $\nu - 1$  with the weight  $v(x)(\cos x - \cos a_1) \dots (\cos x - \cos a_m)$ . The zeros of the polynomial  $S_\nu$  coincide with the nodes  $x_1, x_2, \dots, x_\nu$ ; they should be all distinct and differ from the fixed nodes  $a_1, \dots, a_m$ .

By this result, there exist numbers  $\varepsilon_0, \dots, \varepsilon_{r+1}$  such that for each cosine polynomial  $C_{2r}$  of degree  $2r$  we have

$$\frac{1}{\pi} \int_0^\pi C_{2r}(x) dx = \sum_{k=0}^{r+1} \varepsilon_k C_{2r}(x_k), \quad (15)$$

where  $x_2, x_3, \dots, x_r$  are the zeros of the polynomial  $S_{r-1,\alpha}$  in the interval  $(\alpha, \pi)$ ,  $x_0 = 0$ ,  $x_1 = \alpha$ ,  $x_{r+1} = \pi$ .

Note that, for  $\nu = 1, 2, \dots, r$ , the zeros of the polynomial

$$S_{r-1,x_\nu}(x) = \frac{\sin rx_\nu \sin(r+1)x - \sin(r+1)x_\nu \sin rx}{(\cos x - \cos x_\nu) \sin x} \quad (16)$$

coincide with the zeros of the polynomial  $(\cos x - \cos \alpha)S_{r-1,\alpha}(x)/(\cos x - \cos x_\nu)$ . Thus,

$$S_{r-1,x_\nu}(x) = \mathcal{A}_\nu(\cos x - \cos \alpha)S_{r-1,\alpha}(x)/(\cos x - \cos x_\nu), \quad (17)$$

where  $\mathcal{A}_\nu$  is a constant that does not depend on  $x$ .

It is not difficult to check that, for  $\nu = 1, 2, \dots, r$ , the polynomial (16) satisfies the equations

$$S_{r-1,x_\nu}(x) = \frac{D_r(x - x_\nu) - D_r(x + x_\nu)}{2 \sin x} = 2 \sum_{k=1}^r \frac{\sin kx_\nu \sin kx}{\sin x}, \quad (18)$$

where

$$D_r(x) = 1 + 2 \sum_{k=1}^r \cos kx = \frac{\sin \frac{2r+1}{2}x}{\sin(x/2)}$$

is the Dirichlet kernel.

Using (18), we obtain

$$\frac{1}{\pi} \int_0^\pi S_{r-1,x_\nu}(x)(\sin x)^2 dx = \sin x_\nu, \quad \nu = 1, 2, \dots, r. \quad (19)$$

Using (19), (17) and (18), one can calculate the following coefficients of the quadrature formula (15):

$$\varepsilon_\nu = 1 / \left( 2 \sum_{k=1}^r (\sin kx_\nu)^2 \right) = \frac{\sin x_\nu}{r \sin x_\nu - \sin rx_\nu \cos(r+1)x_\nu}, \quad 1 \leq \nu \leq r. \quad (20)$$

By (14), we have

$$\frac{1}{\pi} \int_0^\pi S_{r-1,\alpha}(x)(1 + \cos x)(\cos \alpha - \cos x) dx = \sin(r+1)\alpha - \sin r\alpha.$$

Using (15) and (12), we obtain from here that

$$\varepsilon_0 = \frac{\sin r\alpha - \sin(r+1)\alpha}{2((r+1)\sin r\alpha - r\sin(r+1)\alpha)}. \quad (21)$$

By (13) and (12) we conclude that

$$\frac{1}{\pi} \int_0^\pi S_{r-1,\alpha}(x)(1 - \cos x)(\cos x - \cos \alpha) dx = (-1)^r (\sin r\alpha + \sin(r+1)\alpha). \quad (22)$$

Formulae (22), (15) and (12) imply

$$\varepsilon_{r+1} = \frac{\sin r\alpha + \sin(r+1)\alpha}{2((r+1)\sin r\alpha + r\sin(r+1)\alpha)}. \quad (23)$$

It is easy to check that

$$\left( \frac{2\pi}{2r+1} - \alpha \right) \varepsilon_r \geq 0 \quad (24)$$

for  $0 < \alpha < \alpha_{2r-1}$ . The statement of the lemma for  $n = 2r - 1$ ,  $r \geq 2$ , now follows from (20), (21), (23) and (24).

Finally, let us consider the case when  $n = 2r - 2$ ,  $r \geq 2$ , and  $0 < \alpha < \pi/r$ . Function (5) can be written in the form  $Q_{(2r+1)/2,\alpha}(x) = \sin(x/2)(\cos x - \cos \alpha)\Theta_{r-1,\alpha}(x)/\sin(\alpha/2)$ , where

$$\Theta_{r-1,\alpha}(x) = \frac{\sin \frac{2r-1}{2}\alpha \sin \frac{2r+1}{2}x - \sin \frac{2r+1}{2}\alpha \sin \frac{2r-1}{2}x}{(\cos x - \cos \alpha) \sin(x/2)} = \frac{(\varphi(x) - \varphi(\alpha)) \sin \frac{2r-1}{2}\alpha \sin \frac{2r-1}{2}x}{(\cos x - \cos \alpha) \sin(x/2)}; \quad (25)$$

here,  $\varphi(x) = (\sin \frac{2r+1}{2}x) / \sin \frac{2r-1}{2}x$ . When  $x$  runs over the intervals  $(0, 2\pi/(2r-1))$ ,  $(2(r-1)\pi/(2r-1), \pi)$  and  $(2k\pi/(2r-1), 2(k+1)\pi/(2r-1))$ ,  $1 \leq k \leq r-2$ , then the values of the function  $\varphi$  run continuously over the intervals  $((2r+1)/(2r-1), -\infty)$ ,  $(+\infty, -1)$  and  $(+\infty, -\infty)$ , respectively. Thus, for  $0 < \alpha < \pi/r$  the polynomial  $\Theta_{r-1,\alpha}$  has exactly  $r-1$  simple zeros  $x_2 < x_3 < \dots < x_r$  in the interval  $(\alpha, \pi)$ . With the help of (13) and (25), repeating the arguments used in the proof of formula (14), we see that

$$\frac{1}{\pi} \int_0^\pi \Theta_{r-1,\alpha}(x) C_{r-1}(x) (\cos \alpha - \cos x) dx = \left( \sin \frac{2r+1}{2}\alpha - \sin \frac{2r-1}{2}\alpha \right) C_{r-1}(0) \quad (26)$$

for all cosine polynomials  $C_{r-1}$  of degree  $r-1$ . Thus, the polynomial  $\Theta_{r-1,\alpha}$  is orthogonal to all cosine polynomials of degree  $r-2$  with the weight  $(1 - \cos x)(\cos x - \cos \alpha)$ . It follows that there exist numbers  $\delta_0, \delta_1, \dots, \delta_r$  such that the quadrature formula

$$\frac{1}{\pi} \int_0^\pi C_{2r-1}(x) dx = \sum_{k=0}^r \delta_k C_{2r-1}(x_k), \quad (27)$$

where  $x_2, x_3, \dots, x_r$  are the zeros of the polynomial  $\Theta_{r-1,\alpha}$  in the interval  $(\alpha, \pi)$ ,  $x_0 = 0$ ,  $x_1 = \alpha$ , is exact for all cosine polynomials  $C_{2r-1}$  of degree  $2r-1$ .

Note that, for  $\nu = 1, 2, \dots, r$ , the polynomial

$$\Theta_{r-1,x_\nu}(x) = \frac{\sin \frac{2r-1}{2}x_\nu \sin \frac{2r+1}{2}x - \sin \frac{2r+1}{2}x_\nu \sin \frac{2r-1}{2}x}{(\cos x - \cos x_\nu) \sin(x/2)} \quad (28)$$

satisfies the equation

$$\Theta_{r-1,x_\nu}(x) = \mathcal{B}_\nu (\cos x - \cos \alpha) \Theta_{r-1,\alpha}(x) / (\cos x - \cos x_\nu), \quad 1 \leq \nu \leq r, \quad (29)$$

where  $\mathcal{B}_\nu$  is a constant that does not depend on  $x$ .

Moreover, the polynomial (28) can be rewritten in the form

$$\Theta_{r-1,x_\nu}(x) = 2 \sum_{k=1}^r \left( \sin \frac{2k-1}{2}x_\nu \sin \frac{2k-1}{2}x \right) / \sin \frac{x}{2}. \quad (30)$$

This implies the equation

$$\frac{1}{\pi} \int_0^\pi \Theta_{r-1,x_\nu}(x) \left( \sin \frac{x}{2} \right)^2 dx = \sin \frac{x_\nu}{2}, \quad 1 \leq \nu \leq r. \quad (31)$$

Formulae (31), (27), (29) and (30) yield

$$\delta_\nu = 1 / \left( 2 \sum_{k=1}^r \left( \sin \frac{2k-1}{2}x_\nu \right)^2 \right) = \frac{2 \sin x_\nu}{2r \sin x_\nu - \sin 2rx_\nu}, \quad 1 \leq \nu \leq r. \quad (32)$$

By (26) we obtain

$$\frac{1}{\pi} \int_0^\pi \Theta_{r-1,x_\nu}(x) (\cos \alpha - \cos x) dx = \sin \frac{2r+1}{2}\alpha - \sin \frac{2r-1}{2}\alpha. \quad (33)$$

Using (33), (27) and (25), we get

$$\delta_0 = \left( \sin \frac{2r-1}{2}\alpha - \sin \frac{2r+1}{2}\alpha \right) / \left( (2r+1) \sin \frac{2r-1}{2}\alpha - (2r-1) \sin \frac{2r+1}{2}\alpha \right). \quad (34)$$

The statement of the lemma for  $n = 2r-2$ ,  $r \geq 2$ , now follows from (32) and (34). This completes the proof of the lemma.  $\square$

**Lemma 2.** *Let  $n$  be a nonnegative integer,  $0 \leq \alpha \leq \alpha_n$  if  $n$  is even and  $0 \leq \alpha < \alpha_n$  if  $n$  is odd. For each polynomial  $\tau_n \in T_n$  we have*

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \tau_n(x)(\cos \alpha - \cos x) dx &= \frac{(\sin \frac{n+3}{2}\alpha - \sin \frac{n+1}{2}\alpha)(1 - \cos \alpha)}{(n+3) \sin \frac{n+1}{2}\alpha - (n+1) \sin \frac{n+3}{2}\alpha} \tau_n(0) \\ &+ \sum_{k=1}^{[(n+1)/2]} g_{n+1}(x_k)(\cos \alpha - \cos x_k)(\tau_n(x_k) + \tau_n(-x_k)), \end{aligned} \quad (35)$$

where  $x_1 < x_2 < \dots < x_{[(n+1)/2]}$  are the zeros of the polynomial (7) in the interval  $(\alpha, \pi]$ , and the numbers  $g_{n+1}(x_k)$ ,  $k = 1, 2, \dots, [(n+1)/2]$ , are defined by equations (11). Moreover, the coefficients  $g_{n+1}(x_k)(\cos \alpha - \cos x_k)$ ,  $k = 1, 2, \dots, [(n-1)/2]$ , are nonnegative, as well as the number  $(\frac{2\pi}{n+2} - \alpha)g_{n+1}(x_{[(n+1)/2]})(\cos \alpha - \cos x_{[(n+1)/2]})$ .

**P r o o f.** For  $0 < \alpha < \alpha_n$ , the statement is a straightforward consequence of Lemma 1. Let  $\tau_n$  be an arbitrary polynomial of degree  $n$ , then the right-hand side of (35) and the coefficients of this quadrature formula tend uniformly to the claimed (bounded) values as  $\alpha \rightarrow 0$ , and the statement of the lemma follows for  $\alpha = 0$ . The case of  $\alpha = \alpha_n$  with even  $n$  can be proved in a similar way. As for the case of odd  $n$ , note that for an odd  $n \geq 3$  we have  $g_{n+1}(x_{[(n+1)/2]})(\cos \alpha - \cos x_{[(n+1)/2]}) = g(\pi)(\cos \alpha + 1) \rightarrow -\infty$  as  $\alpha \rightarrow \alpha_n$ , while  $g_{n+1}(x_{[(n-1)/2]})(\cos \alpha - \cos x_{[(n-1)/2]}) \rightarrow +\infty$  as  $\alpha \rightarrow \alpha_n$ .  $\square$

**P r o o f of the theorem.** The statement of the theorem follows from the fact that for each nonnegative polynomial  $\tau_n$  and each number  $\alpha$  in the interval  $[0, 2\pi/(n+2)]$  we have, by Lemma 2, the inequality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \tau_n(x)(\cos \alpha - \cos x) dx \geq \frac{(\sin \frac{n+3}{2}\alpha - \sin \frac{n+1}{2}\alpha)(1 - \cos \alpha)}{(n+3) \sin \frac{n+1}{2}\alpha - (n+1) \sin \frac{n+3}{2}\alpha} \tau_n(0).$$

This inequality turns into the equality for the polynomial  $\tau_{n,\alpha}$ . This proves the theorem.  $\square$

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